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ZAGREB INDICESOF BOOK GRAPH AND STACKED BOOK GRAPH

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ABSTRACT: Zagreb indices are the parameters defined using sum and product of degrees of vertices, joining an edge, in a graph The roots of this concept come from Chemical graph theory, in recent years lot of work is published on Zagreb indices of standard graphs and graph operations. In this paper we establish the relation for Zagreb indices of book graph and Stacked book graph with its component graphs and also find their corresponding polynomials.

KEYWORDS: Zagreb Indices, Hyper Zagreb Indices, Book Graph, Stacked Book Graph

Classification number: 05C07, 05C35, 05C76

1.1 INTRODUCTION

The concepts of connectivity in Chemical Graph Theory, which define relationships between the structure of a molecule and its properties. One important parameter, topological index, which characterizes molecular graph and remain invariant under graph automorphism are called Zagreb Indices. These parameters introduced by Gutman I [5] are defined by using sum and product of degrees of vertices joining an edge.

Consider a subset of E(G) denoted as $E_{a,b}=\{(u,v) \in E(G) \,/\, d(u) = a \text{ and } d(v) = b\}$. Partitioning the edge set E(G) into disjoint sets $E_{a,b}$ with all possible choices of pairs a,b we can determine the Zagreb and hyper Zagreb indices and their corresponding polynomials for first and second kind. The Zagreb indices**,** hyper Zagreb indices and their corresponding polynomialswe use definitions given by Gutman [5] stated as follows**:**

$$M_1(G) = \sum_{E_{uv} \in E(G)} [d_G(u) + d_G(v)] \qquad M_2(G) = \sum_{uv \in V(G)} [d_G(u) d_G(v)]$$

$$HM_1(G) = \sum_{uv \in E} [d_G(u) + d_G(v)]^2 \quad HM_2(G) = \sum_{uv \in E} [d_G(u) d_G(v)]^2$$

$$M_1(G,x) = \sum_{uv\in E(G)} x^{[d_G(u)+d_G(v)]} \qquad M_2(G,x) = \sum_{uv\in E(G)} x^{[d_G(u)d_G(v)]}$$

$$HM_1(G,x) = \sum_{uv\in E} x^{[d_G(u)+d_G(v)]^2} \qquad HM_2(G,x) = \sum_{uv\in E} x^{[d_G(u)d_G(v)]^2}$$

In this paper we present results for Zagreb indices, hyper Zagreb indices and their polynomials for product graphs B_m, $B_{m,n}$For simplicity of notation we write $d_G(u) =$ a and $d_G(v)$ = b, in all further proofs.

2 ZAGREB INDICES FOR B_m

Book graph B_m is the cross product of star S_{m+1} and path P_2. For $m \geq 3$, we have$|V(B_m)| = 2m + 2$, $|E(B_m)| = 3m + 1$. B_m is a biregular graph with two possible vertex degrees 2 and $m + 1$. We can partition $V(B_m)$ in to two disjoint subsetsV_2 and V_{m+1} as follows.

$V_2 = \{v \in V(B_m)\ /d(v) = 2\}$with $|V_2| = 2m$

$V_{m+1} = \{v \in V(B_m)\ /d(v) = m + 1\}$ with$|V_{m+1}| = 2$

Next, the edge set $E(B_m)$can be partitioned into three disjoint subsets based on the condition that incident vertices belong to V_2 and V_{m+1} as follows.

$E_{2,2} = \{uv \in E(B_m)\ ;\ u,v\epsilon V_2\} with\ |E_{2,2}| = m$

$E_{2,m+1} = \{uv \in E(B_m);\ u\epsilon V_2,\ v\epsilon V_{m+1}\}$ with $|E_{2,m+1}| = 2m$

$E_{m+1,m+1} = \{uv \in E(B_m);\ u,v\epsilon V_{m+1}\}\ with\ |E_{m+1,m+1}| = 1$

Theorem2.1: The first Zagreb indices and their polynomial forB_m are

$$M_1(B_m) = 2[(m+3)^2 - 8] \quad where\ m \geq 3$$

$$M_1(B_m,x) = mx^4 + 2mx^{m+3} + x^{2m+2} \quad where\ m \geq 3$$

Proof: The first Zagreb indices of B_m are

$$M_1(B_m) = \sum_{E_{uv\in E(G)}} [a+b] = \sum_{E_{2,2}} [a+b] + \sum_{E_{m+1,2}} [a+b] + \sum_{E_{m+1,m+1}} [a+b]$$

$$= |E_{2,2}|[2+2] + |E_{2,m+1}|[m+1+2] + |E_{m+1,m+1}|[m+1+m+1]$$

$$= 2m^2 + 12m + 2 = 2[(m+3)^2 - 8]$$

The first Zagreb polynomial of B_m is

$$\mathrm{M}_1(\mathrm{B}_\mathrm{m},x) = \sum_{uv\in E(G)} x^{a+b} = \sum_{E_{2,2}} x^{a+b} + \sum_{\boldsymbol{E}_{\boldsymbol{m+1,2}}} x^{a+b} + \sum_{\boldsymbol{E}_{\boldsymbol{m+1,m+1}}} x^{a+b}$$

$$= |E_{2,2}|x^{[2+2]} + |E_{2,m+1}|x^{[m+1+2]} + |E_{m+1,m+1}|x^{[m+1+m+1]}$$

$$= mx^4 + 2mx^{m+3} + x^{2m+2} \blacksquare$$

Theorem 2.2: The second Zagreb indices and their polynomial forB_m are

$$M_2(B_m) = 5(m+1)^2 - 4 \quad where m \geq 3$$

$$M_2(B_m,x) = mx^4 + 2mx^{2(m+1)} + x^{(m+1)^2} where m \geq 3$$

Proof: The second Zagreb indices forB_m is,

$$M_2(B_m) = \sum_{uv\in V(G)}[a.b] = \sum_{E_{2,2}}[a.b] + \sum_{E_{m+1,2}}[a.b] + \sum_{E_{m+1,m+1}}[a.b]$$

$$= |E_{2,2}|[2\times 2] + |E_{2,m+1}|[(m+1)\times 2] + |E_{m+1,m+1}|[(m+1)^2$$

$$= 4m + 4m^2 + 4m + m^2 + 2m + 1 = 5m^2 + 10m + 1 = 5(m+1)^2 - 4$$

Next second Zagreb polynomial for B_m is

$$M_2(B_m,x) = \sum_{uv\in E(G)} x^{[a.b]} = \sum_{E_{2,2}} x^{[a.b]} + \sum_{E_{m+1,2}} x^{[a.b]} + \sum_{E_{m+1,m+1}} x^{[a.b]}$$

$$= mx^{[2\times 2]} + 2mx^{[(m+1)\times 2]} + x^{[(m+1)\times(m+1)]}$$

$$= mx^4 + 2mx^{2(m+1)} + x^{(m+1)^2} \blacksquare$$

Theorem 2.3 : The first hyper-Zagreb indices and their polynomial of B_m are

$$HM_1(B_m) = 2m[(m+4)^2 + 5] + 4 \quad where\ m \geq 3$$

$$HM_1(B_m,x) = mx^{16} + 2mx^{[m+3]^2} + x^{[2(m+1)]^2}$$

Proof: First hyper Zagreb indices for B_m is

$$HM_1(B_m) = \sum_{uv\in E}[a+b]^2 = \sum_{E_{2,2}}[a+b]^2 + \sum_{E_{m+1,2}}[a+b]^2 + \sum_{E_{m+1,m+1}}[a+b]^2$$

.

$$= m[2+2]^2 + 2m[m+1+2]^2 + 1[m+1+m+1]^2$$

$$= 2m^3 + 16m^2 + 42m + 4HM_1(B_m)$$

$$= 2m[(m+4)^2 + 5] + 4$$

Now the first hyper-Zagreb-polynomial for B_m is

$$HM_1(B_m,x) = \sum_{uv\in E} x^{[a+b]^2} = \sum_{E_{2,2}} x^{[a+b]^2} + \sum_{E_{2,m+1}} x^{[a+b]^2} + \sum_{E_{m+1,m+1}} x^{[a+b]^2}$$

$$= mx^{[2+2]^2} + 2mx^{[m+1+2]^2} + x^{[m+1+m+1]^2}$$

$$HM_1(B_m, x) = mx^{16} + 2mx^{[m+3]^2} + x^{[2(m+1)]^2} \blacksquare$$

Theorem 2.4: The second hyper-Zagreb indices and polynomial of B_m for $m \geq 3$ is

$$HM_2(B_m) = [(m+1)^2 + 4m]^2 - 16m(m-1)$$

$$HM_2(B_m, x) = mx^{16} + 2mx^{(2(m+1))^2} + x^{(m+1)^4}$$

Proof: The second hyper Zagreb indices of B_m for $m \geq 3$ is

$$HM_2(B_m) = \sum_{uv \in E} [a.b]^2 = \sum_{E_{2,2}} [a.b]^2 + \sum_{E_{2,m+1}} [a.b]^2 + \sum_{E_{m+1,m+1}} [a.b]^2$$

$$= m[2 \times 2]^2 + 2m[(m+1) \times 2]^2 + [(m+1) \times (m+1)]^2$$

$$= m^4 + 12m^3 + 22m^2 + 28m + 1$$

$$HM_2(B_m) = [(m+1)^2 + 4m]^2 - 16m(m-1)$$

Now, second hyper-Zagreb polynomial of B_m for $m \geq 3$ is

$$HM_2(B_m, x) = \sum_{uv \in E} x^{[a.b]^2} = \sum_{E_{2,2}} x^{[a.b]^2} + \sum_{E_{m+1,2}} x^{[a.b]^2} + \sum_{E_{m+1,m+1}} x^{[a.b]^2}$$

$$= mx^{[2\times 2]^2} + 2mx^{[(m+1)\times 2]^2} + x^{[(m+1)\times(m+1)]^2}$$

$$= mx^{16} + 2mx^{(2(m+1))^2} + x^{(m+1)^4} \blacksquare$$

3 Zagreb indices for $B_{m,n}$

The graph $B_{m,n}$ is the cross product of S_{m+1} and path P_n for $m \geq 3$ and $n \geq 2$, with the number of vertices $|V(B_{m,n})| = mn + n$ and edges $|E(B_{m,n})| = 2mn - m + n - 1$. We can partition $V(B_{m,n})$ in to four disjoint subsets V_2 , V_3 , V_{m+1} and V_{m+2} as

$$V_2 = \{v \in V(B_{m,n});\ d(v) = 2\}; |V_2| = 2m$$

$V_3 = \{v \in V(B_{m,n});\ d(v) = 3\}; |V_3| = m(n-2)$

$V_{m+1} = \{v \in V(B_{m,n});\ d(v) = m+1\}; |V_{m+1}| = 2$

$V_{m+2} = \{v \in V(B_{m,n});\ d(v) = m+2\}; |V_{m+2}| = n-2$

Next, the edge set $E(B_{m,n})$ can be partitioned into six disjoint subsets based on the degree of incident vertices as follows :

$E_{2,3} = \{uv \in E(B_m);\ u \in V_2,\ v \in V_3\}; |E_{2,3}| = 2m$

$E_{2,m+1} = \{uv \in E(B_m);\ u \in V_2,\ v \in V_{m+1}\};\ |E_{2,m+1}| = 2m$

$E_{m+1,m+2} = \{uv \in E(B_m);\ u \in V_{m+1},\quad v \in V_{m+2}\}; |E_{m+1,m+2}| = 2$

$E_{3,m+2} = \{uv \in E(B_m);\ u \in V_3, v \in V_{m+2}\};\ |E_{3,m+2}| = m(n-2)$

$E_{3,3} = \{uv \in E(B_m);\ u \in V_3,\ v \in V_3\}; |E_{3,3}| = m(n-3)$

$E_{m+2,m+2} = \{uv \in E(B_m);\ u \in V_{m+2},\quad v \in V_{m+2}\}; |E_{m+2,m+2}| = n-3$

We observe that when $n = 3$, $|E_{3,3}| = 0$ and $|E_{m+2,m+2}| = 0$

Theorem 3.1: The first Zagreb indices and their polynomial of $B_{m,n}$ for $m \geq 3$ & $n \geq 3$ are,

$$M_1(B_{m,n}) = \begin{cases} m(mn + 13n - 14) + 2(2n-3) & if m \geq 3 \ and\ n > 3 \\ m^2n + 5m(n+2) + 6 & if\ m \geq 3\ and\ n = 3 \end{cases}$$

$$M_1(B_{m,n}, x) = \begin{cases} (mn-2m)x^{[m+5]} + (n-3)x^{2m+4} + 2mx^{m+3} + 2x^{2m+3} + \\ (n-3)mx^6 + 2mx^5 & if\ n > 3 \\ (mn-2m)x^{[m+5]} + 2mx^{m+3} + 2x^{2m+3} + 2mx^5 & if\ n = 3 \end{cases}$$

Proof: The first Zagreb indices of $B_{m,n}$ is

$$\boldsymbol{M_1(B_{m,n}) = \sum_{uv\in E(G)} [a+b]}$$

$$\boldsymbol{= \sum_{E_{2,3}}[a+b] + \sum_{E_{2,m+1}}[a+b] + \sum_{E_{m+1,m+2}}[a+b] + \sum_{E_{3,m+2}}[a+b]}$$

$$\boldsymbol{+\sum_{E_{3,3}}[a+b] + \sum_{E_{m+2,m+2}}[a+b]}$$

$$= 2m[2+3] + 2m[2+m+1] + 2[m+1+m+2] + (mn-2m)$$

$$\boldsymbol{[3+m+2] + (n-3)m[3+3] + (n-3)[m+ \ 2+m+2]}$$

Case (i) if $n > 3$

$$M_1(B_{m,n}) = m(mn+13n-14) + 2(2n-3)$$

Case (ii) when $\mathbf{n = 3}$ in equation (1), terms $(\mathbf{n-3})\mathbf{6m}$and (n-3) $[\mathbf{2m+4}]$ are equal to zero as $\mathbf{E_{3,3}}$ and $\mathbf{E_{m+2,m+2}}$has no edges giving,

$$M_1(B_{m,n}) = m^2n + 5m(n+2) + 6$$

Now, the first Zagreb polynomial of $B_{m,n}$ is

$$M_1(B_{m,n}, x) = \sum_{uv\in E(G)} x^{[a+b]}$$

$$= \sum_{E_{2,3}} x^{[a+b]} + \sum_{E_{2,m+1}} x^{[a+b]} + \sum_{E_{m+1,m+2}} x^{[a+b]} + \sum_{E_{3,m+2}} x^{[a+b]} + \sum_{E_{3,3}} x^{[a+b]} + \sum_{E_{m+2,m+2}} x^{[a+b]}$$

$$= 2mx^{[2+3]} + 2mx^{[2+m+1]} + 2x^{[m+1+m+2]} + (mn-2m)x^{[3+m+2]} + (n-3)mx^{[3+3]}$$

$$+(n-3)x^{[m+2+m+2]}$$

Case (i) $n > 3$

$$M_1(B_{m,n}, x) = (mn-2m)x^{[m+5]} + (n-3)x^{2m+4} + 2mx^{m+3} + 2x^{2m+3} + (n-3)mx^6 + 2mx^5$$

Case (ii) when $m = n$ in equation (2), $(n-3)mx^6$ and $(\mathrm{n}-3)\mathrm{x}^{(2\mathrm{m}+4)}$ become zero, as $E_{3,3}$ and $E_{m+2,m+2}$ have no edges, giving,

$$M_1(B_{m,n}, x) = (mn - 2m)x^{[m+5]} + 2mx^{m+3} + 2x^{2m+3} + 2mx^5 \blacksquare$$

Theorem 3.2: The second Zagreb indices and their polynomial of $B_{m,n}$ for $m \geq 3 \; \& \; n \geq 3$ are,

$$\mathbf{M_2(B_{m,n})} = \begin{cases} \mathbf{m(4mn + 19n - 3m - 29) + 4(n-2)} \text{ if } \mathbf{m \geq 3 \text{ and } n > 3} \\ \mathbf{(m+2)[3mn+2] + 2(5m+2)} \text{ if } \mathbf{m \geq 3 \text{ and } n = 3} \end{cases}$$

$$M_2(B_{m,n}, x) = \begin{cases} (n-3)x^{(m+2)^2} + (mn-2m)x^{[3m+6]} + 2mx^{2(m+1)} + \\ \quad +2mx^6 + (n-3)mx^9 \, if \; m \geq 3 \; and \; n > 3 \\ (mn-2m)x^{[3m+6]} + 2mx^{2(m+1)} + 2x^{(m^2+3m+2)} + \\ \quad 2mx^6 \, if \; m \geq 3 \; and \; n = 3 \end{cases}$$

Proof : The second Zagreb indices of $B_{m,n}$ are

$$M_2(B_{m,n}) = \sum_{uv \in E(G)} [a.b] = \sum_{E_{2,3}} [a.b] + \sum_{E_{2,m+1}} [a.b] + \sum_{E_{m+1,m+2}} [a.b] + \sum_{E_{3,m+2}} [a.b]$$

$$+ \sum_{E_{3,3}} [a.b] + \sum_{E_{m+2,m+2}} [a.b]$$

$$= 2m[2 \times 3] + 2m[2 \times (m+1)] + 2[(m+1) \times (m+2)] + (mn - 2m)$$

$$+ \boldsymbol{(n-3)m[3 \times 3] + (n-3)[(m+2) \times (m+2)]}$$

Case (i) $\boldsymbol{n} > 3$

$$\boldsymbol{M_2(B_{m,n}) = m(4mn + 19n - 3m - 29) + 4(n - 2)}$$

Case (ii) $\boldsymbol{n = 3}$

Putting$\boldsymbol{n = 3}$ in equation (3),$\boldsymbol{(n-3)6m}$ and $\boldsymbol{(n-3)[2m+4]}$become zero giving.

$$M_2(B_{m,n}) = \boldsymbol{(m+2)[3mn+2] + 2(5m+2)}$$

Next second Zagreb polynomial for stacked book graph $B_{m,n}$ is,

$$M_2(B_{m,n}, x) = \sum_{uv\in E(G)} x^{[a.b]} = \sum_{E_{2,3}} x^{[a.b]} + \sum_{E_{2,m+1}} x^{[a.b]} + \sum_{E_{m+1,m+2}} x^{[a.b]}$$

$$+ \sum_{E_{3,m+2}} x^{[a.b]} + \sum_{E_{3,3}} x^{[a.b]} + \sum_{E_{m+2,m+2}} x^{[a.b]}$$

$$= 2mx^{[2\times 3]} + 2mx^{[2\times(m+1)]} + 2x^{[(m+1)\times(m+2)]} + (mn-2m)x^{[3\times(m+2])}$$
$$+ (n-3)mx^{[3\times 3]} + (n-3)x^{[(m+2)\times(m+2)]}$$

Case 1: $n > 3$

$$M_2(B_{m,n}, x) = (n-3)x^{(m+2)^2} + (mn-2m)x^{[3m+6]} + 2mx^{2(m+1)}$$

$$+2x^{(m^2+3m+2)} + 2mx^6 + (n-3)mx^9$$

Case (ii) when $n = 3$ in above equation,$(n-3)mx^9$and $(n-3)x^{(m+2)^2}$becomes Zero, as $E_{3,3}$ and $E_{m+2,m+2}$ has no edges, giving

$$M_2(B_{m,n}, x) = (mn-2m)x^{[3m+6]} + 2mx^{2(m+1)} + 2x^{(m^2+3m+2)} + 2mx^6 \blacksquare$$

Theorem 3.3: The first hyper Zagreb indices and their polynomial of $B_{m,n}(m \geq 3 \; and \; n \geq 3$ are

$$HM_1(B_{m,n}) = \begin{cases} mn(m^2+14m+77) - m(12m-114) - 2(8n-15) \; if \;\; n > 3 \\ m[(m+5)^2 n + 42] + 18 \quad if \; n = 3 \end{cases}$$

$$\boldsymbol{HM_1(B_{m,n}, x)} = \begin{cases} \boldsymbol{2mx^{25} + 2mx^{(m+3)^2} + 2x^{(2m+3)^2} + (mn-2m)x^{(m+5)^2}} \\ \boldsymbol{+m(n-3)x^{36} + (n-3)x^{[2m+4]^2}} \qquad \boldsymbol{if\; n} > 3 \\ \boldsymbol{2mx^{25} + 2mx^{(m+3)^2} + 2x^{(2m+3)^2} + (mn-2m)x^{(m+5)^2} if \; n = 3} \end{cases}$$

Proof: The first hyper Zagreb indices of $\boldsymbol{B_{m,n}}$

$$\boldsymbol{HM_1(B_{m,n})} = \sum_{\boldsymbol{uv \in E(G)}} [\boldsymbol{a+b}]^{\boldsymbol{2}}$$

$$HM_1(B_{m,n}) = \sum_{E_{2,3}} [a+b]^2 + \sum_{E_{2,m+1}} [a+b]^2 + \sum_{E_{m+1,m+2}} [a+b]^2 + \sum_{E_{3,m+2}} [a+b]^2 +$$

$$\sum_{E_{3,3}} [a+b]^2 + \sum_{E_{m+2,m+2}} [a+b]^2$$

$$= 2m[2+3]^2 + 2m[2+m+1]^2 + 2[m+1+m+2]^2 + (mn-2m)[3+m+2]^2$$

Case (i) :$n > 3$

$$HM_1(B_{m,n}) = mn(m^2+14m+77) - m(12m-114) + 2(8n-15)$$

Case (ii) : when $n = 3$ in equation (5), $(n-3)36m$and $(n-3)[2m+4]$becomes zero as $E_{3,3}$ and $E_{m+2,m+2}$ has no edges, giving

$$HM_1(B_{m,n}) = m[(m+5)^2 n + 42] + 18$$

Next first hyper Zagreb polynomial of $B_{m,n}$ is,

$$HM_1(B_{m,n},x) = \sum_{uv\in E(G)} x^{[a+b]^2}$$

$$= \sum_{E_{2,3}} x^{[a+b]^2} + \sum_{E_{2,m+1}} x^{[a+b]^2} + \sum_{E_{m+1,m+2}} x^{[a+b]^2} + \sum_{E_{3,m+2}} x^{[a+b]^2} + \sum_{E_{3,3}} x^{[a+b]^2}$$

$$+ \sum_{E_{m+2,m+2}} x^{[a+b]^2} + \sum_{E_{m+2,m+2}} x^{[a+b]^2}$$

$$= 2mx^{[2+3]^2} + 2mx^{[2+m+1]^2} + 2x^{[m+1+m+2]^2} + (mn-2m)x^{[3+m+2]^2} + (n-3)$$

$$mx^{[3+3]^2} + (n-3)x^{[2m+4]^2}$$

Case (i) $\boldsymbol{n} > 3$

$$\mathbf{HM_1(B_{m,n},x) = 2mx^{25} + 2mx^{(m+3)^2} + 2x^{(2m+3)^2} + (mn-2m)x^{(m+5)^2} + m(n-3)x^{36}}$$

$$+$$

$$+\mathbf{(n-3)x^{[2m+4]^2}}$$

Case (ii) when $\mathbf{n = 3}$ in equation (6) $\mathbf{(n-3)m}$ and $\mathbf{(n-3)}$ are equal to zero as

$\boldsymbol{E_{3,3}}$ and $\boldsymbol{E_{m+2,m+2}}$ has no edges

$$HM_1(B_{m,n},x) == 2mx^{25} + 2mx^{(m+3)^2} + 2x^{(2m+3)^2} + (mn-2m)x^{(m+5)^2} \blacksquare$$

Theorem 3.4: The second hyper Zagreb indices and polynomial of $B_{m,n}(m \geq 3 \; and \; n = 3)$are

$$\boldsymbol{HM_2(B_{m,n}) = \begin{cases} mn(m^3+9m^2-12m+168)-m(m^3-2m^2-42m+255) \\ \quad +8(2n-5)\ if\ n>3 \\ 9mn(m+2)^2+2m(m^3+m^2-15m+16)+8\ if\ n=3 \end{cases}}$$

$$\boldsymbol{HM_2(B_{m,n},x) = \begin{cases} 2mx^{36}+2mx^{[2m+2)]^2}+2x^{[m^2+3m+2]^2}+(mn-2m)x^{[3m+6]^2} \\ \quad +m(n-3)x^{36}+(n-3)x^{(m^2+4m+4)^2}\ if\ n>3 \\ 2mx^{36}+2mx^{[2m+2)]^2}+2x^{[m^2+3m+2]^2}+(mn-2m)x^{[3m+6]^2} \\ \quad if\ n=3 \end{cases}}$$

Proof: The first hyper Zagreb indices of $\boldsymbol{B_{m,n}}$ is

$$HM_2(B_{m,n}) = \sum_{uv\in E(G)}[a.b]^2 = \sum_{E_{2,3}}[a.b]^2 + \sum_{E_{2,m+1}}[a.b]^2 + \sum_{E_{m+1,m+2}}[a.b]^2 + \sum_{E_{3,m+2}}[a.b]^2$$

$$+\sum_{E_{3,3}}[a.b]^2 + \sum_{E_{m+2,m+2}}[a.b]^2$$

$$= 2m[2\times 3]^2 + 2m[2\times(m+1)]^2 + 2[(m+1)\times(m+2)]^2 + (mn-2m)$$

$$[3\times(m+2)]^2 + (n-3)m[3\times 3]^2 + (n-3)[(m+2)\times(m+2)]^2$$

Case (i) $\boldsymbol{n}>3$

$$\boldsymbol{HM_2(B_{m,n}) = m^4n+17m^3n+60m^2n+149mn-m^4-22m^3-102m^2}$$

$$\boldsymbol{-307m+16n-40}$$

Case (ii) when $n=3$ in equation (7) $(n-3)6m$ and $(n-3)[2m+4]$

are equal to zero as $E_{3,3}$ and $E_{m+2,m+2}$ has no edges

$$HM_2(B_{m,n}) = 9mn(m+2)^2 + 2m(m^3+m^2-15m+16)+8$$

Next second hyper Zagreb polynomial of $B_{m,n}$ is

$$HM_2(B_{m,n},x) = \sum_{uv\in E(G)} x^{[a.b]^2}$$

$$= \sum_{E_{2,3}} x^{[a.b]^2} + \sum_{E_{2,m+1}} x^{[a.b]^2} + \sum_{E_{m+1,m+2}} x^{[a.b]^2} + \sum_{E_{3,m+2}} x^{[a.b]^2} + \sum_{E_{3,3}} x^{[a.b]^2} + \sum_{E_{m+2,m+2}} x^{[a.b]^2}$$

$$= 2mx^{[2\times 3]^2} + 2mx^{[2\times(m+1)]^2} + 2x^{[(m+1)\times(m+2)]^2} + (mn-2m)x^{[3\times(m+2)]^2} + (n-3)mx^{[3+3]^2} + (n-3)x^{[(m+2)\times(m+2)]^2}$$

Case (i) $n > 3$

$$HM_2(B_{m,n},x) = 2mx^{36} + 2mx^{[2m+2)]^2} + 2x^{[m^2+3m+2]^2} + (mn-2m)x^{[3m+6]^2} + m(n-3)x^{36} + (n-3)x^{(m^2+4m+4)^2}$$

Case (ii) when $n = 3$ in equation (8),$(n-3)6m$ and $(n-3)[2m+4]$becomeszero as $E_{3,3}$ and $E_{m+2,m+2}$ has no edges, giving

$$HM_2(B_{m,n},x) = 2mx^{36} + 2mx^{[2m+2)]^2} + 2x^{[m^2+3m+2]^2} + (mn-2m)x^{[3m+6]^2}$$

4 Results :

In the paper of Cangul et al [8] have given values for M_1, M_2 and we found the results for hyper Zagreb indices for star and path graph as given below :

Sl.no	Zagreab	S_{m+1}	P_n
1	M_1	m^2+m	4n – 6
2	M_2	m^2	4n-8
3	HM_1	$m(m+1)^2$	8(2n-3)
4	HM_2	m^3	16n-30

Expressing the results for product graphs in terms of their component graph parameters we observe that

1. $\boldsymbol{if m \geq 3 \text{ and } n > 3}$

$$M_1(B_{m,n}) = m(mn + 13n - 14) + 2(2n - 3)$$

$$= \mathbf{n(m^2 + m) + (3m + 1)(4n - 6) + 4}$$

$$= \mathbf{n\ M_1(S_{m+1}) + (3m + 1)M_1(P_n) + 4m}$$

$$HM_2(B_{m,n}) = mn(m^3 + 9m^2 - 12m + 168) - m(m^3 - 2m^2 - 42m + 255) +$$

$$+8(2n - 5)$$

$$= m^3[mn + 9n - m + 2] + (10m + 1)(16n - 30) + 8n + 35$$

$$= [mn + 9n - m + 2]\ \mathbf{HM_2(S_{m+1})} + (10m+1)\mathbf{HM_2(P_n)} + 8n + 35$$

2. $if\ m \geq 3\ and\ n = 3$

$$M_1(B_{m,n}) = m^2n + 5m(n+2) + 6 = n(m^2+m) + m(4n-6) + 16m + 6$$

$$= n\, M_1(S_{m+1}) + (m+1)\, M_1(P_n) + 16m$$

$$M_2(B_{m,n}) = (m+2)[3mn+2] + 2(5m+2)$$

$$= 3nm^2 + \frac{3m}{2}(4n-8) + 8(3m+1)$$

$$= 3nM_2(S_{m+1}) + \frac{3m}{2}M_2(P_n) + 8(3m+1)$$

$$HM_1(B_{m,n}) = m[(m+5)^2n + 42] + 18$$

$$= nm(m+1)^2 + m(16n-24) + 8m^2n + 8mn + 66m + 18$$

$$= n\, HM_1(S_{m+1}) + mHM_1(P_n) + 8m^2n + 8mn + 66m + 18$$

$$9mn(m+2)^2 + 2m(m^3 + m^2 - 15m + 16) + 8$$

$$HM_2(B_{m,n}) = 9mn(m+2)^2 + 2m(m^3 + m^2 - 15m + 16)$$

$$= m^3[9n + 2m + 2] + (16n - 30)\left[\frac{9}{4}m\right](m+1) - \frac{144}{4}m^2 + \frac{119}{4}m + 8$$

$$= [9n + 2m + 2]\, HM_2(S_{m+1}) + \left[\frac{9}{4}m\right](m+1)HM_2(P_n)$$

$$- \frac{144}{4}m^2 + \frac{119}{4}m + 8$$

5 Conclusion:

Thus,from above results we can conclude the Zagreb indices of product graph $\boldsymbol{B_{mn}}$ with respective Zagreb indices of its factor graphs satisfy Vizing conjecture like results, where $\boldsymbol{\alpha}$ represents various Zagreb indices

$$\alpha(B_{m,n}) = \alpha(S_{m+1} \times P_n) \geq |V(P_n)|\, \alpha(S_{m+1}) + |V(S_{m+1})|\, \alpha(P_n)$$

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